Weight spectrum of codes associated with the Grassmannian $G(3,7)$

Krishna Kaipa, Harish K. Pillai

Abstract

In this paper we consider the problem of determining the weight spectrum of q-ary codes $C(3,m)$ associated with Grassmann varieties $G(3,m)$. For $m = 6$ this was done in [1]. We derive a formula for the weight of a codeword of $C(3,m)$, in terms of certain varieties associated with alternating trilinear forms on $\mathbb{F}_q^m$. The classification of such forms under the action of the general linear group $GL(m, \mathbb{F}_q)$ is the other component that is required to calculate the spectrum of $C(3,m)$. For $m = 7$, we explicitly determine the varieties mentioned above. The classification problem for alternating 3-forms on $\mathbb{F}^7$ was solved in [2] which we then use to determine the spectrum of $C(3,7)$.

I. INTRODUCTION

GRASSMANN codes are linear codes associated with the Grassmann variety $G(\ell, m)$ of ℓ dimensional subspaces of an m dimensional vector space $\mathbb{F}^m$, where $\mathbb{F}$ is a finite field with q elements. They were first studied by Ryan [3] for $q = 2$, and Nogin [4] for general q. These codes are conveniently described using the correspondence between non-degenerate $[n, k]_q$ linear codes on one hand and non-degenerate $[n, k]$ projective systems on the other hand [5]. A non-degenerate $[n, k]$ projective system is simply a collection of n points in projective space $\mathbb{P}^{k-1}$ satisfying the condition that no hyperplane of $\mathbb{P}^{k-1}$ contains all the n points under consideration.

The projective system used to define the Grassman codes $C(\ell, m)$ is given by the classical Plücker embedding of $G(\ell, m)$ in $\mathbb{P}(\wedge^\ell \mathbb{F}^m)$, the projective ℓ-th exterior power of $\mathbb{F}^m$. Thus the parameters n and k of the codes $C(\ell, m)$ are :

$$n = |G(\ell, m)| = \frac{(q^m - 1)(q^{m-1} - 1)\dots(q^{m-\ell+1} - 1)}{(q^\ell - 1)(q^{\ell-1} - 1)\dots(q - 1)}$$
$$k = \binom{m}{\ell} \tag{1}$$

We briefly recall the Plücker embedding. To an ℓ dimensional subspace Λ of $\mathbb{F}^m$ we assign the wedge product $v_1 \wedge v_2 \wedge \cdots \wedge v_\ell$ where $\{v_1, \dots, v_\ell\}$ is an arbitrary basis of Λ. The expression $v_1 \wedge \cdots \wedge v_\ell$ considered as an element of $\mathbb{P}(\wedge^\ell \mathbb{F}^m)$ is independent of the choice of basis. This defines a one-one map of $G(\ell, m)$ in $\mathbb{P}(\wedge^\ell \mathbb{F}^m)$. The image of this map is a non-singular variety defined by the Plücker relations. Let $\{e_1, \dots, e_m\}$ be a basis of $\mathbb{F}^m$ and let $\mathcal{I}(\ell, m)$ denote the set of multi-indices:

$$\mathcal{I}(\ell, m) = \{(i_1, \cdots, i_\ell) \mid 1 \le i_1 < i_2 < \cdots < i_\ell \le m\}$$

Then $\{e_I \mid I \in \mathcal{I}(\ell, m)\}$ is a basis of $\wedge^\ell \mathbb{F}^m$. In terms of this basis, the Plücker image of the ℓ dimensional subspace Λ is given as

$$\sum_{I \in \mathcal{I}(\ell, m)} p_I \, e_I$$

where the k homogeneous coordinates p_I are called the Plücker coordinates of Λ. A hyperplane $\mathcal{H}$ in $\mathbb{P}(\wedge^\ell \mathbb{F}^m)$ is given by a linear equation: $\sum_{I \in \mathcal{I}(\ell, m)} c_I p_I = 0$. It is clear that if $c_I \ne 0$, then the ℓ-plane

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having basis $\{e_i \,|\, i \in I\}$ does not lie in $\mathcal{H}$. This shows that the Plücker embedding is non-degenerate.

Assigning some order to the n points $\{P_1, \ldots, P_n\}$ of $G(\ell, m)$, and also to the k elements of $\mathcal{I}(\ell, m)$, we form the $k \times n$ matrix M whose entries M_{ij} are given by the i-th Plücker coordinate of P_j. The matrix M is the generator matrix of the code $C(\ell, m)$. The non-degeneracy condition implies that the matrix has full rank. If we left-multiply M by a message-word (a row vector $(a_1, \ldots, a_k)$ of length k), we obtain a codeword (a row vector $(b_1, \ldots, b_n)$ of length n) in a one-one manner. Thus M generates a linear code $\mathbb{F}^k \hookrightarrow \mathbb{F}^n$. Observe that the row-span of M is the space of codewords. The entries b_i of the codeword are equal to the values at P_i of the functional on $\wedge^\ell \mathbb{F}^m$, given by $\sum_{I \in \mathcal{I}(\ell, m)} a_I e^I$ where $\{e^I = e^{i_1} \wedge \cdots \wedge e^{i_\ell} \,|\, I \in \mathcal{I}(\ell, m)\}$ is the dual basis to $\{e_I \,|\, I \in \mathcal{I}(\ell, m)\}$. Thus we see that the message-words correspond to elements of $\wedge^\ell(\mathbb{F}^m)^*$, the space of functionals on $\wedge^\ell(\mathbb{F}^m)$. There is a bijective correspondence between hyperplanes $\mathcal{H}$ in $\mathbb{P}(\wedge^\ell \mathbb{F}^m)$ and points $\omega_\mathcal{H}$ of the projective space of non-zero message-words $\mathbb{P}(\wedge^\ell(\mathbb{F}^m)^*)$. In this correspondence, the kernel of the functional $\omega_\mathcal{H}$ is precisely $\mathcal{H}$. We may also think of elements of $\mathbb{P}(\wedge^\ell(\mathbb{F}^m)^*)$ as the projective space of alternating ℓ-multilinear functions (or ℓ-forms) on $\mathbb{F}^m$.

The weight of a codeword corresponding to $\omega_\mathcal{H}$ (i.e its Hamming norm) is simply the number of points $P_i, 1 \le i \le n$ not lying on $\mathcal{H}$. By abuse of notation, we often refer to $\omega_\mathcal{H}$ as the codeword. Consider the function $\omega \mapsto \mathrm{wt}(\omega)$ from non-zero codewords to positive integers. The image of this function, together with the number of pre-images for each integer in the image, is called the spectrum of the code $C(\ell, m)$. The weight of a non-zero codeword ω only depends on its projective class. Therefore, for determining the spectrum of $C(\ell, m)$, it suffices to consider only the projective space of codewords.

The weight spectrum of codes $C(2, m)$ for all m, and the weight spectrum of $C(3, 6)$ were determined by Nogin in [4] and [1] respectively. The organization of the article is as follows. In section II we introduce what we call the weight varieties associated to a 3-form, and derive a formula for the weight of a codeword of $C(3, m)$ in terms of the cardinalities of these varieties. The calculation of the spectrum of $C(3, m)$ requires us to determine the possible values of these weights, as well as the number of codewords having each of these weights. This weight classification of codewords is facilitated by the classification of projective 3-forms on $\mathbb{F}^m$ under the action of the projective linear group $PGL(m, \mathbb{F})$. In section III, we obtain this classification for $m = 7$, by a minor modification of the results of the authors of [2]. In section IV we determine the weight varieties (and their cardinalities), of representative codewords of this classification, and calculate the spectrum of $C(3, 7)$.

II. A FORMULA FOR THE WEIGHT OF A 3-FORM

We derive a formula for the weight of a codeword of the code $C(3, m)$. The following notation will be used in this section. V denotes the vector space $\mathbb{F}^m$. For any set $A \subset V \setminus \{0\}$ which satisfies $c \cdot A \subset A$ for all non-zero scalars c, we use the notation $\mathbb{P}A$ to denote the projectivization of A. For a finite set S, $|S|$ denotes its cardinality. The cardinality of the general linear group $GL(m, \mathbb{F})$ will be denoted by $[m]_q$:

$$[m]_q = q^{m(m-1)/2}(q^m - 1)(q^{m-1} - 1) \cdots (q - 1)$$

Given a codeword of $C(3, m)$, let ω be the corresponding 3-form on $\mathbb{F}^m$, and let $\mathcal{H}_\omega$ be the corresponding hyperplane of $\mathbb{P}(\wedge^3 \mathbb{F}^m)$ as described above. The weight of the codeword ω is

$$\mathrm{wt}(\omega) = |\{P_i \,:\, 1 \le i \le n, P_i \notin \mathcal{H}_\omega\}|$$

We will frequently use the following observation: the total number of ordered bases of all 3-dimensional subspaces of V represented by the $P_i \notin \mathcal{H}_\omega$ put together, is $[3]_q \mathrm{wt}(\omega)$.

Definition II.1. *The group $GL(m, \mathbb{F})$ acts on 3-forms by taking a 3-form ω to the 3-form $g \cdot \omega$ defined by*

$$(g \cdot \omega)(v_1, v_2, v_3) = \omega(gv_1, gv_2, gv_3)$$

For a 3-form ω on $\mathbb{F}^m$ we define $\text{Aut}(\omega)$ *to be the group:*

$$\text{Aut}(\omega) = \{g \in GL(m, \mathbb{F}) \,|\, g \cdot \omega = \omega\}$$

A. Weight of a degenerate 3-form

We consider the map $\phi_\omega : V \to \wedge^2 V^*$ sending $v \mapsto \iota_v \omega$ where ι_v is the operation of interior multiplication defined by:

$$\langle \iota_v \omega, \beta \rangle = \langle \omega, v \wedge \beta \rangle, \quad \forall \beta \in \wedge^2 V$$

Here $\langle\, , \rangle$ is the pairing between $\wedge^j V^*$ and $\wedge^j V$ for each j.

Definition II.2. *We say that the 3-form ω is non-degenerate if $\ker(\phi_\omega) = \{0\}$.*

If ω is degenerate, let $\ker(\phi_\omega)$ be r-dimensional. We pick a basis $\{e_1, \cdots e_m\}$ of V such that that $\{e_{m-r+1}, \cdots, e_m\}$ is a basis for $\ker(\phi_\omega)$. Let W denote the span of $\{e_1, \cdots, e_{m-r}\}$. Let $\tilde\omega$ denote the restriction of the form ω to W. Since $W \cap \ker(\phi_\omega) = \{0\}$, it is clear that $\tilde\omega$ is a non-degenerate 3-form on W. Thus $\tilde\omega$ can be thought of as a codeword in $C(3, m-r)$.

Proposition II.3. $\text{wt}(\omega) = q^{3r} \text{wt}(\tilde\omega)$ and $|\text{Aut}(\omega)| = |\text{Aut}(\tilde\omega)| \, [r]_q \, q^{r(m-r)}$

Proof We have:

$$[3]_q \cdot \text{wt}(\omega) = |\{[v_1, v_2, v_3] \,:\, \langle \omega, v_1 \wedge v_2 \wedge v_3 \rangle \neq 0\}| \tag{2}$$

where $[v_1, v_2, v_3]$ denotes a $m \times 3$ matrix with columns v_1, v_2, and v_3. Since the expression for ω in terms of the dual basis $\{e^1, \ldots, e^m\}$ is independent of $e^{m-r+1}, \ldots, e^m$, the last r rows of the matrix $[v_1, v_2, v_3]$ are arbitrary. Moreover if $[u_1, u_2, u_3]$ is the submatrix formed by the first $m - r$ rows then :

$$[3]_q \cdot \text{wt}(\omega) = q^{3r} \, |\{[u_1, u_2, u_3] \,:\, \langle \tilde\omega, u_1 \wedge u_2 \wedge u_3 \rangle \neq 0\}|$$
$$= q^{3r} \cdot [3]_q \, \text{wt}(\tilde\omega)$$

If $g \in \text{Aut}(\omega)$ then the equation $\omega(gv_1, gv_2, gv_3) = \omega(v_1, v_2, v_3)$ implies that $\iota_{gv_1}\omega$ is zero if and only if $\iota_{v_1}\omega = 0$. Thus $\text{Aut}(\omega)$ carries $\ker(\phi_\omega)$ to itself. Therefore, with respect to the basis $\{e_1, \cdots, e_m\}$, we can write $g = \left(\begin{smallmatrix} \tilde g & 0 \\ k & h \end{smallmatrix}\right)$. Such a matrix g is in $\text{Aut}(\omega)$ if and only if $\tilde g \in \text{Aut}(\tilde\omega)$, $h \in GL(r, \mathbb{F})$, and k an arbitrary $(m - r) \times r$ matrix. Thus the cardinality of $\text{Aut}(\omega)$ is $[r]_q \, q^{(m-r)r}$ times the cardinality of $\text{Aut}(\tilde\omega)$. $\qquad\square$

The proposition shows that in order to calculate the weights of codewords of $C(3, m)$, it is enough to know only the weights of non-degenerate codewords of $C(3, \tilde m)$ for $\tilde m \leq m$. The cardinality of $\text{Aut}(\omega)$ is useful in determining the number of codewords having a given weight (i.e the spectrum).

B. Weight varieties of a non-degenerate 3-form

Let V be an m-dimensional vector space over an arbitrary field F. Given a 2-form $\lambda \in \wedge^2 V^*$, we define certain quantities $\text{Pf}_k(\lambda) \in \wedge^{2k} V^*$, for each $k \geq 1$ which we call the k-th Pfaffian of λ. Let $\text{Pf}_0(\lambda) = 1$. We define $\text{Pf}_k(\lambda) \in \wedge^{2k} V^*$ inductively by requiring:

$$\iota_v \lambda \wedge \text{Pf}_{k-1}(\lambda) = \iota_v \text{Pf}_k(\lambda), \quad \forall v \in V \tag{3}$$

This $\text{Pf}_k(\lambda)$ generalizes the forms $\frac{\lambda^k}{k!} = \frac{1}{k!}(\lambda \wedge \cdots \wedge \lambda)$, which are used over the fields of real and complex numbers, to fields with arbitrary characteristic. We recall the following standard diagonalization theorem (see [6], section $XV.8$) for 2-forms on V. The rank of a 2-form λ is the rank of the matrix whose (i, j)-th entry is $\lambda(e_i, e_j)$ for any basis $\{e_1, \cdots, e_m\}$ of V. The rank is an even integer $2r$, and one can always pick a basis of V such that the associated matrix is block diagonal with r blocks consisting of the 2×2 matrix $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and zeros elsewhere.

Proposition II.4 (k-th Pfaffians of a 2-form).

1) *Given a $\lambda \in \wedge^2 V^*$, $\lambda \neq 0$, for each $k \geq 1$ there is a unique element $\mathrm{Pf}_k(\lambda) \in \wedge^{2k} V^*$ satisfying* (3).

2)
$$\mathrm{Pf}_k(\lambda_1 + \lambda_2) = \sum_{j=0}^{k} \mathrm{Pf}_j(\lambda_1) \wedge \mathrm{Pf}_{k-j}(\lambda_2) \tag{4}$$

3) *The unique integer $2r$ such that $\mathrm{Pf}_r(\lambda) \neq 0$ and $\mathrm{Pf}_{r+1}(\lambda) = 0$, is the rank of λ.*

Proof We assume inductively that the first assertion of the proposition holds for $1 \leq j \leq k-1$. The uniqueness of $\mathrm{Pf}_k(\lambda)$ follows from the fact that any form α is determined completely by the forms $\{\iota_v \alpha \,|\, v \in V\}$. As for existence, we consider the function

$$f(v_1, v_2, \ldots, v_{2k}) = \langle \iota_{v_1} \lambda \wedge \mathrm{Pf}_{k-1}(\lambda), v_2 \wedge \cdots \wedge v_{2k} \rangle$$

The function f is clearly multilinear. It is also alternating in the variables $v_2, \ldots, v_{2k}$. In order to prove that f is a $2k$-form on V, it suffices to show that $f(v_1, v_1, v_3, \ldots, v_{2k}) = 0$:

$$\begin{aligned}
f(v_1, v_1, v_3, \ldots, v_{2k}) &= \langle \iota_{v_1} \lambda \wedge \mathrm{Pf}_{k-1}(\lambda), v_1 \wedge v_3 \cdots \wedge v_{2k} \rangle \\
&= \langle -\iota_{v_1} \lambda \wedge \iota_{v_1} \mathrm{Pf}_{k-1}(\lambda), v_3 \wedge \cdots \wedge v_{2k} \rangle \\
&= \langle -\iota_{v_1} \lambda \wedge \iota_{v_1} \lambda \wedge \mathrm{Pf}_{k-2}(\lambda), v_3 \wedge \cdots \wedge v_{2k} \rangle \\
&= 0 \quad (\text{because } \iota_{v_1} \lambda \wedge \iota_{v_1} \lambda = 0)
\end{aligned}$$

It also follows from the definition of f that $\iota_{v_1} f = \iota_{v_1} \lambda \wedge \mathrm{Pf}_{k-1}(\lambda)$ thus proving that $\mathrm{Pf}_k(\lambda) = f$.

The second assertion easily follows from the defining equation (3) and induction. To prove the third assertion, we observe that if $\mathrm{Pf}_j(\lambda) = 0$, then by (3), $\mathrm{Pf}_i(\lambda) = 0$ for all $i \geq j$. Since $\mathrm{Pf}_1(\lambda) = \lambda \neq 0$, there is a unique integer 2ρ satisfying $\mathrm{Pf}_\rho(\lambda) \neq 0$ and $\mathrm{Pf}_{\rho+1}(\lambda) = 0$.

Using a special basis $\{e_1, \cdots, e_n\}$ of F^m as in the diagonalization theorem mentioned before the proposition, we write

$$\lambda = e^1 \wedge e^2 + e^3 \wedge e^4 + \cdots + e^{2r-1} \wedge e^{2r}$$

where $2r$ is the rank of λ and $\{e^1, \cdots, e^m\}$ is the dual basis. Using this expansion of λ in (4), we calculate all the k-th Pfaffians of λ, and find that $\rho = r$.

Remarks If $\mathrm{char}(F)$ does not divide $k!$, then by uniqueness $\mathrm{Pf}_k(\lambda)$ is simply $\frac{\lambda^k}{k!}$. The equations $\mathrm{Pf}_2(\lambda) = 0$ are the Plücker equations defining decomposable elements of $\mathbb{P}(\wedge^2 V^*)$, or in other words the Plücker embedding of $G(2, V^*)$ in $\mathbb{P}(\wedge^2 V^*)$. Given a skew symmetric $2k \times 2k$ matrix A with entries in F, (with diagonal terms required to be zero if $\mathrm{char}(F) = 2$), we can associate a 2-form to it by $\lambda = \sum_{i<j} A_{ij} e^i \wedge e^j$ where $\{e^1, \cdots, e^{2k}\}$ is the dual basis to the standard basis of F^{2k}. Then $\mathrm{Pf}_k(\lambda)$ equals $e^1 \wedge \cdots \wedge e^{2k}$ times a scalar $\mathrm{Pf}(A)$ (whose square equals $\det(A)$) known as the Pfaffian of the matrix A ([6], section $XV.9$). We also mention the fact $\mathrm{Pf}_k(g \cdot \lambda) = g \cdot \mathrm{Pf}_k(\lambda)$ for all $g \in GL(m, F)$, which can easily be proved by induction starting from the case $k = 1$ and the defining equation (3).

Definition II.5. *Given a non-degenerate 3-form ω on $\mathbb{F}^m$, the k-th weight variety of ω is the subvariety of $\mathbb{P}^{m-1}$ given by:*

$$X_k(\omega) = \mathbb{P}\{x \in \mathbb{F}^m \setminus \{0\} \,|\, \mathrm{Pf}_{k+1}(\iota_x \omega) = 0\}$$

We have

$$\varnothing = X_0(\omega) \subset X_1(\omega) \subset \cdots \subset X_{\lfloor \frac{m-1}{2} \rfloor}(\omega) = \mathbb{P}^{m-1}$$

We will need Nogin's result on spectrum of $C(2, m)$:

Theorem II.6 (Nogin [4]). *The weight of a codeword in $C(2, m)$ depends only on the rank of its associated 2-form ω. If rank(ω) is $2r$, where $1 \leq r \leq \lfloor \frac{m}{2} \rfloor$, then:*

$$\text{wt}(\omega) = q^{2(m-r-1)} \frac{q^{2r} - 1}{q^2 - 1}$$

For each of these $\lfloor \frac{m}{2} \rfloor$ weights, the number of codewords of $C(2, m)$ of that weight is also determined in [2]. We do not need it here.

Theorem II.7. *Given a non-degenerate 3-form ω on $\mathbb{F}^m$, let*

$$n_i := |X_i(\omega)| - |X_{i-1}(\omega)|$$

The weight wt*(ω) is given by:*

$$\text{wt}(\omega) = \frac{q^{2m-4}}{(q^2 - 1)(1 + q + q^2)} \sum_{i=1}^{\lfloor \frac{m-1}{2} \rfloor} n_i (1 - q^{-2i}) \tag{5}$$

Proof For any $v_1 \neq 0$, let $\{e_1, e_2, \ldots, e_m\}$ be a basis of V such that $e_1 = v_1$. Let W denote the subspace generated by $\{e_2, \ldots, e_m\}$, and let $\pi : V \to W$ be the projection on the last $m - 1$ coordinates. Let ω_{v_1} be the 2-form on W obtained by restricting $\iota_{v_1}\omega$ to W. A pair of vectors $v_2, v_3 \in V$ satisfy $\langle \iota_{v_1}\omega, v_2 \wedge v_3 \rangle \neq 0$ if and only if $\langle \omega_{v_1}, \pi(v_2) \wedge \pi(v_3) \rangle \neq 0$. Since the first components of v_2, v_3 are arbitrary, the cardinality of such pairs $\{v_2, v_3\}$ is:

$$q^2 \, [2]_q \, \text{wt}(\omega_{v_1})$$

We thus have:

$$
\begin{aligned}
[3]_q \cdot \text{wt}(\omega) &= |\{[v_1, v_2, v_3] : \langle \omega, v_1 \wedge v_2 \wedge v_3 \rangle \neq 0\}| \\
&= \sum_{v_1 \neq 0} |\{[v_2, v_3] : \langle \iota_{v_1}\omega, v_2 \wedge v_3 \rangle \neq 0\}| \\
&= q^2 \, [2]_q \sum_{v_1 \neq 0} \text{wt}(\omega_{v_1})
\end{aligned}
$$

In the sum over all $v_1 \neq 0$, there are $(q-1)n_i$ terms for which the line through v_1 is in $X_i(\omega) \setminus X_{i-1}(\omega)$. For such a v_1, ω_{v_1} has rank $2i$ as a 2-form on W. By Theorem II.6

$$\text{wt}(\omega_{v_1}) = q^{2(m-1-i-1)} \frac{q^{2i} - 1}{q^2 - 1}$$

Substituting this expression for wt(ω_{v_1}) above, we get

$$
\begin{aligned}
\text{wt}(\omega) &= \frac{q^2 [2]_q (q-1)}{[3]_q} \sum_{i=1}^{\lfloor \frac{m-1}{2} \rfloor} n_i \frac{q^{2i} - 1}{q^2 - 1} q^{2m-4-2i} \\
&= \frac{q^{2m-4}}{(q^2 - 1)(1 + q + q^2)} \sum_{i=1}^{\lfloor \frac{m-1}{2} \rfloor} n_i (1 - q^{-2i})
\end{aligned}
$$

$\square$

For later use, we specialize formula (5) to the cases $m = 6, 7$. For the case $m = 6$, we use $n_1 + n_2 = |\mathbb{P}^5|$ in (5) to get

$$\text{wt}(\omega) = q^4 \left[(q^5 + q^3 + q^2 + 1) - \frac{n_1}{1 + q + q^2} \right] \tag{6}$$

For the case $m = 7$, we use $n_1 + n_2 + n_3 = |\mathbb{P}^6|$ in (5) to get

$$
\begin{aligned}
\text{wt}(\omega) = q^4 \Big[(q^8 + q^6 + q^5 + q^4 + q^3 + q^2 + 1) \\
- \frac{n_2 + n_1(1 + q^2)}{1 + q + q^2} \Big] \tag{7}
\end{aligned}
$$

C. The variety $X_2(\omega)$ of a non-degenerate 3-form on $\mathbb{F}^7$

Let $V = \mathbb{F}^7$ and $\omega \in \wedge^3 V^*$. We show that the variety $X_2(\omega) \subset \mathbb{P}^6$ is a quadric hypersurface given by the vanishing of an explicitly determined quadratic form Q_ω on V.

Let η be a basis of the 1-dimensional space $\wedge^7 V^*$, and let $H_\eta : \wedge^6 V^* \to V$ be the isomorphism defined by:

$$\alpha \wedge \beta = \langle \beta, H_\eta(\alpha) \rangle \, \eta, \quad \forall \alpha \in \wedge^6 V^*, \beta \in V^* \tag{8}$$

Let $x \in V \setminus \{0\}$ and let $\alpha = \mathrm{Pf}_3(\iota_x \omega)$. We claim that $H_\eta(\alpha)$ is a scalar multiple of x. Since α is trilinear in x, the scalar multiple is a quadratic form $Q(x)$. Pick $y \in \wedge^6 V$ so that $\langle \eta, x \wedge y \rangle = 1$. For any $\beta \in V^*$ with $\langle \beta, x \rangle = 0$, we will show that $\langle \beta, H_\eta(\alpha) \rangle = 0$, thus proving that $H_\eta(\alpha)$ is a scalar multiple of x. By the definition of H_η, and choice of y it follows that:

$$\langle \beta, H_\eta(\alpha) \rangle = \langle \langle \beta, H_\eta(\alpha) \rangle \eta, x \wedge y \rangle = \langle \alpha \wedge \beta, x \wedge y \rangle$$
$$= \langle \iota_x(\alpha \wedge \beta), y \rangle = \langle \iota_x \alpha \wedge \beta, y \rangle$$

where in the last equality, we have used the fact that $\iota_x \beta = \langle \beta, x \rangle = 0$. Using the defining property (3) of the 3-Pfaffian, and the fact that $\alpha = \mathrm{Pf}_3(\iota_x \omega)$, it follows that $\iota_x \alpha = 0$, and hence that

$$H_\eta(\mathrm{Pf}_3(\iota_x \omega)) = Q(x) \cdot x$$

The variety $X_2(\omega) = \mathbb{P}\{x \in \mathbb{F}^7 \setminus \{0\} \,|\, \mathrm{Pf}_3(\iota_x \omega) = 0\}$ can now be expressed as:

$$X_2(\omega) = \mathbb{P}\{x \in \mathbb{F}^7 \setminus \{0\} \,|\, Q(x) = 0\}$$

A different choice $\eta' = a\eta$ for the basis vector of $\wedge^7 V^*$ (where a is a nonzero scalar) gives an isomorphism $H_{\eta'} = a^{-1} H_\eta$, and hence to the quadratic form $a^{-1} Q(x)$. Since the zero locus of $Q(x)$ and $a^{-1} Q(x)$ is the same, the variety $X_2(\omega)$ does not depend on the choice of η. We summarize the above discussion:

Theorem II.8. *The variety $X_2(\omega) \subset \mathbb{P}^6$ associated with a non-degenerate 3-form ω on $\mathbb{F}^7$ is a quadric hypersurface given by the vanishing of a quadratic form Q_ω on $\mathbb{F}^7$. The form Q_ω is defined by*

$$H_\eta(\mathrm{Pf}_3(\iota_x \omega)) = Q_\omega(x) \cdot x \quad \forall x \in \mathbb{F}^7 \tag{9}$$

where $H_\eta : \wedge^6 (\mathbb{F}^7)^* \to \mathbb{F}^7$ is the linear isomorphism defined in (8)

Remark: Applying H_η to the equation:

$$6 \, \mathrm{Pf}_3(\iota_x \omega) = (\iota_x \omega)^3 = \iota_x(\omega \wedge \iota_x \omega \wedge \iota_x \omega)$$

we get $6 \, Q_\omega(x) \eta = \omega \wedge \iota_x \omega \wedge \iota_x \omega$. If $\mathrm{char}(\mathbb{F}) \neq 2, 3$, this relation defines $Q(x)$. If $\mathrm{char}(\mathbb{F}) = 2, 3$ we have to use (9) to define $Q(x)$.

III. $PGL(7, \mathbb{F})$ CLASSIFICATION OF 3-FORMS ON $\mathbb{F}^7$

Let G denote $GL(7, \mathbb{F})$. We recall the action of G on 3-forms as given in definition II.1.

Definition III.1. *We say nonzero 3-forms ω_1 and ω_2 are projectively equivalent if there is a $g \in G$ and a non-zero scalar c such that $g \cdot \omega_1 = c \, \omega_2$. We denote this equivalence relation as $\omega_1 \sim \omega_2$.*

Equivalently $\omega_1 \sim \omega_2$, if their projective classes are in the same orbit under the $\bar{G} = PGL(7, \mathbb{F})$ on $\mathbb{P}(\wedge^3 (\mathbb{F}^7)^*)$ induced by the G-action on $\wedge^3 V^*$. Here $PGL(7, \mathbb{F}) = GL(7, \mathbb{F})/\mathbb{F}^*$ is the projective linear group and $\mathbb{F}^*$ denotes the subgroup in $GL(7, \mathbb{F})$ of scalar matrices.

Let $\omega_1 \sim \omega_2$ with $g \cdot \omega_1 = c \omega_2$. Then $\omega_2(v_1, v_2, v_3) = 0$ if and only if $\omega_1(g \cdot v_1, g \cdot v_2, g \cdot v_3) = 0$. The formula (2) then implies that $\mathrm{wt}(\omega_1) = \mathrm{wt}(\omega_2)$. Therefore, in order to determine the possible weights of all codewords, it suffices to restrict the classification to projective equivalence classes of 3-forms. There is also a notion of linear equivalence obtained by requiring $c = 1$ in Definition III.1. The linear equivalence

classes of non-zero 3-forms on $\mathbb{F}^7$ and their cardinalities were determined by Cohen and Helminck [2]. By grouping together linear classes which have representatives differing by a scalar multiple, we obtain the projective equivalence classes. The sum of the cardinalities of the linear classes in each such group is equal to $q-1$ times the cardinality of the corresponding projective class. We thus obtain the following theorem:

Theorem III.2. *There are eleven projective equivalence classes in $\mathbb{P}(\wedge^3(\mathbb{F}^7)^*)$ with representatives and cardinalities as given below*

$$
\begin{aligned}
\omega_1 &= e^{123} \\
\omega_2 &= e^1 \wedge (e^{23} + e^{45}) \\
\omega_3 &= e^{123} + e^{456} \\
\omega_4 &= e^{123} + e^{345} + e^{561} \\
\omega_{5a} &= e^1 \wedge (e^{23} + e^{45}) + e^6 \wedge (e^{24} + se^{35}), \text{ if char } \mathbb{F} \neq 2 \\
\omega_{5b} &= e^1 \wedge (e^{23} + e^{45}) + e^6 \wedge (e^{24} + se^{35} + e^{45}), \text{ char } \mathbb{F} = 2 \\
\omega_6 &= e^1 \wedge (e^{23} + e^{45}) + e^{267} \\
\omega_7 &= e^1 \wedge (e^{23} + e^{45} + e^{67}) \\
\omega_8 &= e^1 \wedge (e^{23} + e^{57}) + e^6 \wedge (e^{27} + e^{45}) \\
\omega_9 &= e^1 \wedge (e^{23} + e^{45} + e^{67}) + e^{246} \\
\omega_{10} &= e^{123} + e^{456} + e^7 \wedge (e^{14} + e^{25} + e^{36}) \\
\omega_{11a} &= \omega_{5a} + e^{167}, \text{ if char } \mathbb{F} \neq 2 \\
\omega_{11b} &= \omega_{5b} + e^{167}, \text{ char } \mathbb{F} = 2
\end{aligned}
\tag{10}
$$

$$
\begin{aligned}
N_1 &= (q^7 - 1)(q^5 - 1)(q^2 - q + 1)/(q-1)^2 \\
N_2 &= q^2(q^7 - 1)(q^5 - 1)(q^4 + q^2 + 1)(q^3 - 1)/(q-1)^2 \\
N_3 &= \tfrac{1}{2}q^9(q^7 - 1)(q^5 - 1)(q^3 + 1)(q^2 + 1)/(q-1) \\
N_4 &= q^4(q^7 - 1)(q^6 - 1)(q^5 - 1)(q^4 - 1)/(q-1)^2 \\
N_5 &= \tfrac{1}{2}q^9(q^7 - 1)(q^5 - 1)(q^3 - 1)(q+1) \\
N_6 &= \tfrac{1}{2}q^9(q^7 - 1)(q^6 - 1)(q^5 - 1)(q^3 - 1)(q^2 + 1)/(q-1)^2 \\
N_7 &= q^6(q^7 - 1)(q^5 - 1)(q^2 + q + 1) \\
N_8 &= q^{11}(q^7 - 1)(q^6 - 1)(q^5 - 1)(q^2 + q + 1)(q^2 + 1) \\
N_9 &= q^6(q^7 - 1)(q^6 - 1)(q^5 - 1)(q^4 - 1)/(q-1) \\
N_{10} &= q^{15}(q^7 - 1)(q^5 - 1)(q^4 - 1)(q^3 - 1) \\
N_{11} &= \tfrac{1}{2}q^9(q^7 - 1)(q^6 - 1)(q^5 - 1)(q^3 - 1)
\end{aligned}
\tag{11}
$$

Remarks on Theorem III.2 The notation e^{123} denotes $e^1 \wedge e^2 \wedge e^3$. The symbol s above denotes a fixed element of $\mathbb{F}$ satisfying the condition that s is not a square if char $\mathbb{F} \neq 2$, and that s is not of the form $a(a+1)$ in case char $\mathbb{F} = 2$. The number N_i denotes the cardinality of the projective equivalence class of ω_i. The linear equivalence class of a 3-form ω has cardinality $|G|/|\text{Aut}(\omega)|$. For each $\omega \in \{\omega_1, \cdots, \omega_{11}\}$ except ω_{10}, and any non-zero scalar c, the form $c\omega$ is linearly equivalent to ω. To see this we just observe that if $g \in G$ sends e_2, e_5, e_7 to $e_2/c, e_5/c, e_7/c$ and fixes the other basic vectors, then $g \cdot \omega_j = \omega_j$ for $j = 1, 2, 3, 4, 5, 7, 9, 11$, whereas if $g \in G$ sends e_1, e_6 to $e_1/c, e_6/c$ and fixes the other basic vectors, then $g \cdot \omega_j = \omega_j$ for $j = 6, 8$. Thus the cardinalities N_j of the projective classes ω_j, for $j \neq 10$, are obtained by dividing the cardinalities of the linear equivalence classes by $q - 1$. (The forms $\{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6, \omega_7, \omega_8, \omega_9, \omega_{11}\}$ are denoted in [2] by $\{f_1, f_2, f_3, f_4, f_{10}, f_5, f_8, f_7, f_6, f_{11}\}$ respectively.) The set of all 3-forms projectively equivalent to ω_{10} consists of three or one linear equivalence

classes according to whether 3 divides $q - 1$ or not (denoted in tables 1,2 of [2] by $f_9, f_{12,\mu}, f_{12,\mu^2}$ in the former case and just f_9 in the latter case). However the sum of the cardinalities of these linear equivalence classes is always $(q - 1)N_{10}$. The computation of $|\mathrm{Aut}(\omega_{10})|$ is not elementary. The authors of [2] use the fact that $\mathrm{Aut}(\omega_{10})$ is (possibly upto a cyclic group of order 3) the automorphism group of the split algebra of Cayley octonions over $\mathbb{F}$. The latter group is the Chevalley exceptional group $G_2(\mathbb{F})$ of order $q^6(q^6 - 1)(q^2 - 1)$. The number $(q - 1)N_1$ as calculated in [2] (the first entry of table 2) has a typographical error. The denominator in that expression should be $q^2 - 1$ instead of $(q - 1)^2$.

Note that $N_1, \cdots, N_5$ can also be calculated using Proposition II.3. If $\omega \in \{\omega_1, \cdots, \omega_5\}$ and ω' denotes the restriction of ω to $\mathbb{F}^6$ (the span of $\{e_1, \cdots, e_6\}$), and $\tilde{\omega}$ and r are as in the proposition II.3. Then, we have:

$$|\mathrm{Aut}(\omega')| = |\mathrm{Aut}(\tilde{\omega})|\,[r - 1]_q\,q^{(r-1)(m-r)}$$
$$|\mathrm{Aut}(\omega)| = |\mathrm{Aut}(\tilde{\omega})|\,[r]_q\,q^{r(m-r)}$$

Therefore for $1 \leq j \leq 5$, $N_j = \frac{1}{q-1}|G|/|\mathrm{Aut}(\omega_j)|$ equals:

$$N_j = \frac{[7]_q\,[r - 1]_q}{[6]_q\,[r]_q\,q^{7-r}}\,\frac{A_j}{q - 1}$$

where $A_j = |GL(6, \mathbb{F})|/|\mathrm{Aut}(\omega_j)|$ as calculated by Nogin in [1].

IV. WEIGHT CLASSIFICATION OF 3-FORMS ON $\mathbb{F}^7$

The weights of the non-degenerate forms ω_i, $i > 5$ can be determined from formula (7) once the cardinalities of the varieties $X_1(\omega_i)$ and $X_2(\omega_i)$ are known. We begin with $X_1(\omega)$. We recall that $X_1(\omega) = \mathbb{P}\{x \in \mathbb{F}^7 \,|\, \mathrm{Pf}_2(\iota_x\omega) = 0\}$

Proposition IV.1. *The varieties $X_1(\omega_i)$ and their cardinalities for $i > 5$ are:*

$$X_1(\omega_6) = \mathbb{P}^2 \cup_{\mathbb{P}^0} \mathbb{P}^2, \quad n_1(\omega_6) = 1 + 2q + 2q^2$$
$$X_1(\omega_7) = \mathbb{P}^5, \quad n_1(\omega_7) = |\mathbb{P}^5|$$
$$X_1(\omega_8) = \mathbb{P}^1, \quad n_1(\omega_8) = 1 + q$$
$$X_1(\omega_9) = \mathbb{P}^2, \quad n_1(\omega_9) = 1 + q + q^2$$
$$X_1(\omega_{10}) = \varnothing, \quad n_1(\omega_{10}) = 0$$
$$X_1(\omega_{11}) = \mathbb{P}^0, \quad n_1(\omega_{11}) = 1$$

Proof Let $x = \sum_{j=1}^7 x_j e_j$. By Proposition II.4, we have

$$\mathrm{Pf}_2(\iota_x\omega) = \sum_{j=1}^7 x_j^2\,\mathrm{Pf}_2(\iota_{e_j}\omega) + \sum_{i<j} x_i x_j\,(\iota_{e_i}\omega) \wedge (\iota_{e_j}\omega) \tag{12}$$

We begin with ω_6 and evaluate $\mathrm{Pf}_2(\iota_x\omega_6)$ using the above formula (12). We find that the coefficients of e^{2345} and e^{3167} are x_1^2 and x_2^2 respectively. Setting these equal to zero we get:

$$\mathrm{Pf}_2(\iota_x\omega_6)_{|x_1=x_2=0} = e^{12} \wedge (x_4 e^5 - x_5 e^4) \wedge (x_7 e^6 - x_6 e^7)$$

Therefore:

$$X_1(\omega_6) = \{x_1 = x_2 = 0\} \cap (\{x_4 = x_5 = 0\} \cup \{x_6 = x_7 = 0\})$$
$$= \mathbb{P}\{e_3, e_6, e_7\} \cup_{\mathbb{P}\{e_3\}} \mathbb{P}\{e_3, e_4, e_5\} \simeq \mathbb{P}^2 \cup_{\mathbb{P}^0} \mathbb{P}^2$$

Next we consider $\mathrm{Pf}_2(\iota_x\omega_7)$. The coefficient of e^{2345} is x_1^2, moreover x_1 divides $\mathrm{Pf}_2(\iota_x\omega_7)$. Therefore:

$$X_1(\omega_7) = \{x_1 = 0\} \simeq \mathbb{P}^5$$

For $\mathrm{Pf}_2(\iota_x\omega_8)$, the coefficients of $e^{2357}, e^{1367}, e^{1467}$ and e^{2457} are $x_1^2, -x_2^2, -x_5^2$ and x_6^2 respectively. Setting x_1, x_2, x_5 and x_6 to zero, $\mathrm{Pf}_2(\iota_x\omega_8)$ reduces to $(x_3x_4 + x_7^2)e^{1256}$. Therefore

$$X_1(\omega_8) = \{x_1 = x_2 = x_5 = x_6 = x_3x_4 + x_7^2 = 0\} \simeq \mathbb{P}^1$$

The map $(t, s) \mapsto (0, 0, t^2, -s^2, 0, 0, ts)$ establishes an isomorphism between $\mathbb{P}^1$ and $X_1(\omega_8)$.

In $\mathrm{Pf}_2(\iota_x\omega_9)$, the coefficients of $e^{2345}, e^{1346}, e^{1526}$ and e^{1247} are $x_1^2, -x_2^2, x_4^2$ and $-x_6^2$ respectively. Setting x_1, x_2, x_4 and x_6 to zero, $\mathrm{Pf}_2(\iota_x\omega_9)$ reduces to 0. Therefore

$$X_1(\omega_9) = \{x_1 = x_2 = x_4 = x_6 = 0\} \simeq \mathbb{P}^2$$

Next we consider $\iota_x\omega_{10}$ which equals

$$x_1(e^{47} + e^{23}) + x_2(e^{57} + e^{31}) + x_3(e^{67} + e^{12})$$
$$+x_4(e^{71} + e^{56}) + x_5(e^{72} + e^{64}) + x_6(e^{73} + e^{45})$$
$$+x_7(e^{14} + e^{25} + e^{36})$$

The coefficients of $e^{4723}, e^{5731}, e^{6712}, e^{7156}, e^{7264}, e^{7345}$ and e^{1425} in $\mathrm{Pf}_2(\iota_x\omega_{10})$ are equal to $x_1^2, x_2^2, x_3^2, x_4^2, x_5^2, x_6^2$ and x_7^2 respectively. Setting these equal to zero we get:

$$X_1(\omega_{10}) = \varnothing$$

Let μ be one if $\mathrm{char}(\mathbb{F}) = 2$ and zero otherwise. We calculate $\iota_x\omega_{11}$ to be:

$$x_1(e^{23} + e^{45} + e^{67}) + x_2(e^{31} + e^{46}) + x_3(e^{12} + se^{56})$$
$$+x_4(e^{51} + e^{62} + \mu e^{56}) + x_5(e^{14} + se^{63} + \mu e^{64})$$
$$+x_6(e^{24} + se^{35} + \mu e^{45} + e^{71}) + x_7 e^{16}$$

The coefficients in $\mathrm{Pf}_2(\iota_x\omega_{11})$ of e^{2367} and e^{2471} are x_1^2 and x_6^2 respectively. Thus x_1, x_6 must be zero. The coefficients of e^{3146} and e^{1256} are $x_2^2 - sx_5^2 - \mu\, x_2 x_5$ and $sx_3^2 - x_4^2 + \mu\, x_3 x_4$ respectively. By definition of $s \in \mathbb{F}$, the last two quadratic forms are irreducible and hence, x_2, x_3, x_4 and x_5 must all be zero for $\mathrm{Pf}_2(\iota_x\omega_{11})$ to vanish. Therefore:

$$X_1(\omega_{11}) = \{x_1 = \cdots = x_6 = 0\} \simeq \mathbb{P}^0$$

$\square$

We now compute the varieties $X_2(\omega)$ and their cardinalities.

Proposition IV.2. *The varieties $X_2(\omega_i)$ and their cardinalities for $i > 5$ are:*

$$X_2(\omega_6) = \mathbb{P}^5 \cup_{\mathbb{P}^4} \mathbb{P}^5, \quad |X_2(\omega_6)| = 2|\mathbb{P}^5| - |\mathbb{P}^4|$$
$$X_2(\omega_7) = \mathbb{P}^5, \quad |X_2(\omega_7)| = |\mathbb{P}^5|$$
$$X_2(\omega_8) = (\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{F}^3)\, \mathrm{II}\, \mathbb{P}^2, \quad |X_2(\omega_8)| = q^3|\mathbb{P}^1|^2 + |\mathbb{P}^2|$$
$$X_2(\omega_9) = \mathbb{P}^5, \quad |X_2(\omega_9)| = |\mathbb{P}^5|$$
$$X_2(\omega_{10}) = \overline{SL(\mathbb{F}^4)/Sp(\mathbb{F}^4)}, \quad |X_2(\omega_{10})| = |\mathbb{P}^5|$$
$$X_2(\omega_{11}) = \mathbb{P}^4, \quad |X_2(\omega_{11})| = |\mathbb{P}^4|,$$

Proof Let $x = \sum_{j=1}^7 x_j e_j$. By Proposition II.4, we have

$$\mathrm{Pf}_3(\iota_x\omega) = \sum_{j=1}^7 x_j^3\, \mathrm{Pf}_3(\iota_{e_j}\omega) + \tag{13}$$
$$\sum_{i<j} \left[x_i^2 x_j\, \mathrm{Pf}_2(\iota_{e_i}\omega) \wedge (\iota_{e_j}\omega) + x_i x_j^2\, (\iota_{e_i}\omega) \wedge \mathrm{Pf}_2(\iota_{e_j}\omega) \right]$$

Let $* : \wedge^6(\mathbb{F}^7)^* \to \mathbb{F}^7$ be the linear isomorphism defined by:

$$*(e^1 \wedge \cdots \wedge e^{i-1} \wedge e^{i+1} \wedge \cdots \wedge e^7) = (-1)^{i-1} e_i$$

By Proposition II.8, there is a unique quadratic form Q_ω on V such that

$$*(\mathrm{Pf}_3(\iota_x \omega)) = Q_\omega(x) \cdot x$$

The variety $X_2(\omega)$ is the zero locus of Q_ω. To determine Q_ω we expand $\mathrm{Pf}_3(\iota_x \omega)$ using (13). The quadratic form $Q_\omega(x)$ is simply the coefficient of e^{234567} divided by x_1.

We calculate $\mathrm{Pf}_3(\iota_x \omega_6)$ using the above formula (13) and identify $Q_{\omega_6}(x) = x_1 x_2$. Therefore

$$X_2(\omega_6) = \{x_1 x_2 = 0\} \simeq \mathbb{P}^5 \cup_{\mathbb{P}^4} \mathbb{P}^5$$

and $|X_2(\omega_6)| = 2|\mathbb{P}^5| - |\mathbb{P}^4|$.

Next we consider $\mathrm{Pf}_3(\iota_x \omega_7)$ and calculate $Q_{\omega_7}(x) = x_1^2$. Therefore:

$$X_2(\omega_7) = X_1(\omega_7) = \{x_1 = 0\} \simeq \mathbb{P}^5$$

Calculating $\mathrm{Pf}_3(\iota_x \omega_8)$, we get $Q_{\omega_8}(x) = x_1 x_5 - x_2 x_6$. The variety $X_2(\omega_8)$ is the disjoint union of the the subvariety for which at least one of x_1, x_2, x_5, x_6 is non-zero, with the subvariety for which x_1, x_2, x_5, x_6 are all zero. The first subvariety is immediately seen to be $(\mathbb{P}^1 \times \mathbb{P}^1) \times \mathbb{F}^3$ by the Segre embedding, and the second subvariety is $\mathbb{P}^2$.

$$X_2(\omega_8) = \{x_1 x_5 - x_2 x_6 = 0\} \simeq (\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{F}^3) \amalg \mathbb{P}^2$$

and hence $|X_2(\omega_8)| = q^3 |\mathbb{P}^1|^2 + |\mathbb{P}^2|$.

Next we consider $\mathrm{Pf}_3(\iota_x \omega_9)$ and calculate $Q_{\omega_9}(x) = x_1^2$. Therefore:

$$X_2(\omega_9) = \{x_1 = 0\} \simeq \mathbb{P}^5$$

Calculating $\mathrm{Pf}_3(\iota_x \omega_{10})$, we get $Q_{\omega_{10}}(x) = x_1 x_4 + x_2 x_5 + x_3 x_6 - x_7^2$.
The cardinality of $X_2(\omega_{10})$ can easily be calculated to be $|\mathbb{P}^5|$ (but $X_2(\omega_{10})$ is not $\mathbb{P}^5$, see below). We give a description of $X_2(\omega_{10})$ and use it to compute the cardinality. Writing $\mathbb{P}^6 = \mathbb{F}^6 \amalg \mathbb{P}^5$ where the affine part $\mathbb{F}^6$ corresponds to $x_7 = 1$, and $\mathbb{P}^5$ is the hyperplane at infinity $x_7 = 0$, the variety $X_2(\omega_{10}) \subset \mathbb{P}^6$ is a disjoint union $V_0 \amalg V_1$, where $V_1 = X_2(\omega_{10}) \cap \mathbb{F}^6$ is the affine part, and $V_0 = X_2(\omega_{10}) \cap \mathbb{P}^5$ is the part at infinity. $X_2(\omega_{10})$ is thus the projective closure of the affine variety V_1. Comparing with the Plücker relation $\mathrm{Pf}_2(\alpha) = 0$ defining the Grassmannian $G(2,4)$ of 2-forms of rank 2 on $\mathbb{F}^4$, we see V_0 is isomorphic to $G(2,4)$. The variety V_1 is isomorphic to the variety:

$$\{\alpha \in \wedge^2(\mathbb{F}^4)^* \,|\, \mathrm{Pf}_2(\alpha) = e^{1234}\}$$

The formula $g \cdot \mathrm{Pf}_2(\alpha) = \mathrm{Pf}_2(g \cdot \alpha)$ implies that any α with $\mathrm{Pf}_2(\alpha) = e^{1234}$ is of the form $g \cdot (e^{12} + e^{34})$ for a $g \in SL(4, \mathbb{F})$ uniquely determined upto left multiplication by an element of $Sp(\mathbb{F}^4)$. Thus $V_1 = SL(4, \mathbb{F})/Sp(\mathbb{F}^4)$, and hence $X_2(\omega_{10})$ is the projective closure of the affine variety $SL(\mathbb{F}^4)/Sp(\mathbb{F}^4))$.

$$X_2(\omega_{10}) = \{x_1 x_4 + x_2 x_5 + x_3 x_6 = x_7^2\} \simeq \overline{SL(\mathbb{F}^4)/Sp(\mathbb{F}^4)}$$

The cardinality $|X_2(\omega_{10})|$ equals $|V_0| + |V_1|$:

$$(q^4 + q^3 + 2q^2 + q + 1) + \frac{q^6(q^4-1)(q^3-1)(q^2-1)}{q^4(q^4-1)(q^2-1)} = |\mathbb{P}^5|$$

In the case of $\mathrm{Pf}_3(\iota_x \omega_{11})$ we get $Q_{\omega_{11a}}(x) = x_1^2 - s\,x_6^2$, and $Q_{\omega_{11b}}(x) = x_1^2 + s\,x_6^2 + x_1 x_6$. By definition of $s \in \mathbb{F}$ in the cases $\mathrm{char}(\mathbb{F}) \neq 2$, $\mathrm{char}(\mathbb{F}) = 2$ the quadratic forms $Q_{11a}(x), Q_{11b}(x)$ respectively, are irreducible. Therefore $Q_{11}(x)$ vanishes if and only if x_1 and x_6 both vanish:

$$X_2(\omega_{11}) = \{x_1 = x_6 = 0\} \simeq \mathbb{P}^4$$

□

Theorem IV.3. *The weights of $\omega_1, \cdots, \omega_{11}$ are:*

$$
\begin{aligned}
\mathrm{wt}(\omega_1) &= q^{12} \\
\mathrm{wt}(\omega_2) &= q^{12} + q^{10} \\
\mathrm{wt}(\omega_3) &= q^{12} + q^{10} + q^9 - q^7 \\
\mathrm{wt}(\omega_4) &= q^{12} + q^{10} + q^9 \\
\mathrm{wt}(\omega_5) &= q^{12} + q^{10} + q^9 + q^7 \\
\mathrm{wt}(\omega_6) &= q^{12} + q^{10} + q^9 + q^8 - q^7 \\
\mathrm{wt}(\omega_7) &= q^{12} + q^{10} + q^8 \\
\mathrm{wt}(\omega_8) &= q^{12} + q^{10} + q^9 + q^8 \\
\mathrm{wt}(\omega_9) &= q^{12} + q^{10} + q^9 + q^8 \\
\mathrm{wt}(\omega_{10}) &= q^{12} + q^{10} + q^9 + q^8 + q^6 \\
\mathrm{wt}(\omega_{11}) &= q^{12} + q^{10} + q^9 + q^8 + q^7
\end{aligned}
\tag{14}
$$

Proof By Proposition II.3, the weight of a degenerate form ω_i, $1 \le i \le 5$ is q^3 times the weight of ω_i viewed as a 3-form on $\mathbb{F}^6 = \mathrm{span}$ of $\{e_1, \cdots, e_6\}$. The latter weights were determined in [1]. Multiplying them with q^3 we get the weights of $\omega_1, \cdots, \omega_5$.

For the nondegenerate forms $\omega_6, \cdots, \omega_{11}$, we use the formula (7) with $n_2(\omega) + n_1(\omega) = |X_2(\omega)|$:

$$
\begin{aligned}
\mathrm{wt}(\omega_i) = q^{12} + q^{10} &+ q^9 + q^8 + q^7 + q^6 + q^4 \\
&- q^4 \left(\frac{|X_2(\omega_i)| + q^2 |X_1(\omega_i)|}{1 + q + q^2} \right)
\end{aligned}
$$

The quantities $|X_1(\omega_i)|$ and $|X_2(\omega_i)|$ have been computed in Propositions IV.1 and IV.2. Substituting these in the above equation we get the weights of $\omega_6, \cdots, \omega_{11}$. □

We observe that the weights of ω_8 and ω_9 are equal. So we conclude:

Theorem IV.4. *The spectrum of the Grassmann code $C(3,7)$ has ten distinct weights:*

$$
\{\mathrm{wt}(\omega_1), \cdots, \mathrm{wt}(\omega_8), \mathrm{wt}(\omega_{10}), \mathrm{wt}(\omega_{11})\}
$$

where $\mathrm{wt}(\omega_i)$ are given in (14). The number of codewords with weight $\mathrm{wt}(\omega_i)$ for $i = 1 \cdots 8, 10, 11$ are $q - 1$ times $N_1, N_2, \cdots, N_7, N_8 + N_9, N_{10}, N_{11}$ respectively, where N_i are given in (11)

Let μ_1 denote the number of codewords of the dual code $C(3,7)^\perp$ which have weight 1. The non-degeneracy of $C(3,7)$ implies that $\mu_1 = 0$. The MacWilliams identities can be used to express μ_1 in terms of $N_1, \cdots N_{11}$ and the Krawtchouk polynomial $K_1(x) = (q-1)n - qx$ ([7] p.129, or [5] p.19). We get:

$$
0 = \frac{K_1(0)}{q-1} + \sum_{i=1}^{11} N_i K_1(\mathrm{wt}(\omega_i))
$$

Simplifying this equation we get:

$$
\sum_{i=1}^{11} N_i \, \mathrm{wt}(\omega_i) = q^{34} \, n
\tag{15}
$$

Using $\sum_{i=1}^{11} N_i = \frac{q^{35}-1}{q-1} = |\mathbb{P}^{34}|$, and $n = |G(3,7)|$, we can rewrite the above equation as

$$
\frac{1}{|\mathbb{P}^{34}|} \sum_{i=1}^{11} N_i \, \mathrm{wt}(\omega_i) = |G(3,7)| \left(1 - \frac{|\mathbb{P}^{33}|}{|\mathbb{P}^{34}|} \right)
$$

which has the interpretation that the average weight of a $C(3, 7)$ codeword equals $|G(3, 7)|$ times the fraction of points of $\mathbb{P}^{34}$ not lying on a fixed hyperplane. We verified this identity on a computer algebra system by evaluating the left hand side of (15) using the weights from (14) and the N_i's from (11).

V. CONCLUDING REMARKS: SPECTRUM OF $C(3, m)$, $m > 7$

Theorem II.7 allows us to calculate the weight of a non-degenerate codeword ω of $C(3, m)$ in terms of the cardinalities of its weight varieties $X_1(\omega), \cdots, X_{\lfloor \frac{m-1}{2} \rfloor}(\omega)$. Proposition II.3 reduces the calculation of weights of degenerate codewords of $C(3, m)$ to that of non-degenerate codewords of $C(3, \tilde{m})$ for $\tilde{m} < m$. The image of the function $\omega \mapsto \mathrm{wt}(\omega)$ from non-zero codewords to positive integers, and the number of pre-images of each integer in its image, is the spectrum of $C(3, m)$. Since the number of non-zero codewords is $q^{\binom{m}{3}} - 1$ is large, it is not feasible to evaluate the weight function for all codewords. The method proposed here for $m \leq 7$ is to use the fact that projectively or linearly equivalent codewords have the same weights, to evaluate the weights only on the projective or linear equivalence classes. Let $\nu := \binom{m}{3} - m^2$. We note that $\nu < 0$ iff $m \leq 8$. Let $m > 8$ and let $\gamma(m, q)$ denote the number of linear equivalence classes of $C(3, m)$ codewords. If $\omega_1, \cdots, \omega_{\gamma(m,q)}$ are representatives of these equivalence classes then:

$$\sum_{i=1}^{\gamma(m,q)} \frac{|GL(m, \mathbb{F})|}{|\mathrm{Aut}(\omega_i)|} = q^{\binom{m}{3}} - 1$$

Since $|GL(m, \mathbb{F})|/|\mathrm{Aut}(\omega_i)| \leq |GL(m, \mathbb{F})|$, and $|GL(m, \mathbb{F})| = q^{m^2} + O(q^{m^2 - 1})$ we get

$$\gamma(m, q) \geq q^\nu + O(q^{\nu - 1}) \quad \text{for } m > 8$$

The number of projective equivalence classes is thus greater than $q^{\nu - 1} + O(q^{\nu - 2})$. Although the number of distinct weights is in general less than the number of projective classes, we believe that the former will still be bounded below by polynomial function of q for any fixed $m > 8$.

The problem of calculating the spectrum of the code $C(3, 8)$ on the other hand is much more tractable. By Proposition II.3 we need determine only the weights of non-degenerate 3-forms. Noui [8] has shown that there are 13 linear equivalence classes of non-degenerate 3-forms over $\bar{\mathbb{F}}^8$ where $\bar{\mathbb{F}}$ is an algebraic closure of $\mathbb{F}$. Let $\omega_{8,1}, \cdots, \omega_{8,13}$ (in the notation of [8], [9]) be representative 3-forms for these classes. Distinct linear equivalence classes over $\mathbb{F}$ may turn out to be linearly equivalent over $\bar{\mathbb{F}}$. Following the method used by Cohen and Helminck [2] for $m = 7$, once the groups $\mathrm{Aut}(\omega_{8,j}) \subset GL(8, \bar{\mathbb{F}})$ are known, the methods of Galois cohomology can be used to determine the classes $\omega_{8,j}$ which split into multiple classes when going from $\bar{\mathbb{F}}$ to $\mathbb{F}$. This program is partially carried out by Noui and Midoune [9] (Corollary 2) for the first 6 forms $\omega_{8,1}, \cdots, \omega_{8,6}$. Under the restriction $\mathrm{char}(\mathbb{F}) \neq 2, 3$, they show explicitly that these 6 classes over $\bar{\mathbb{F}}$ yield 9 classes over $\mathbb{F}$. As future work one can complete this program, and use it to fully determine the spectrum of $C(3, 8)$.

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